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A CONSTRUCTIVE METHOD OF ESTABLISHING THE VALIDITY OF THE THEORY OF SYSTEMS WITH NON-RETAINING CONSTRAINTS*

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The formal axiomatic approach to establishing the validity of the theory of constrained systems has obvious disadvantages: the source of the initial axioms (such as the *Befreiungsprinzip* and the conditions for constraints to be ideal) remains unclear. A constructive method is proposed for establishing the validity of the main principles of the dynamics of unilaterally constrained systems (including systems with collisions). The idea of the method is related to the analysis of physical methods for realising constraints (stiff systems, anisotropic viscosity, and apparent additional masses). This approach yields simple equations of motion, suitable for the entire time interval and more accurately incorporating the actual dynamics. Several problems of the mechanics of oscillatory systems with collisions are solved by the method. In particular, conditions are determined for the stability of periodic oscillatory modes and a study is made of the evolution of motion with inelastic collisions when the coefficient of restitution is close to unity. Total integrability is established and a qualitative analysis is presented of the problem of parabolic billiards in a uniform force field.

1. Systems with bilateral constraints.

Let $T(x^{\cdot}, x)$ be the kinetic energy and $V(x)$ the potential energy of a holonomic system with n degrees of freedom. The dynamics of the system is described by the Lagrange equations with Lagrangian $L = T - V$, which is assumed to be a smooth function of x and $x^{\cdot}$. Let $f(x)$ be a smooth regular function ($df \neq 0$ at points where $f = 0$). If we take $f(x) = 0$ as the equation of a constraint imposed on the system, the motion of this "constrained" holonomic system with $n - 1$ degrees of freedom is described by the Lagrange equations with multiplier λ:

$$(L_{x^{\cdot}}')^{\cdot} - L_x' = \lambda f_x', \quad f(x) = 0 \tag{1.1}$$

Eqs.(1.1) are usually derived using the D'Alembert-Lagrange principle. In a formal, axiomatic setting, the validity of the D'Alembert-Lagrange principle is derived from the *Befreiungsprinzip* and the axiom that the constraints are ideal. From the physical standpoint, it seems more promising to adopt a constructive approach to establishing the validity of the dynamics of constrained systems, based on an analysis of various concrete realizations of constraints (stiff systems, anisotropic viscosity, and apparent additional masses).

To illustrate the possibilities, let us consider the dynamics of an "unconstrained" holonomic system with n degrees of freedom, with kinetic energy $T_N = T + \alpha (f^{\cdot})^2/2$, and potential energy $V_N = V + \beta f^2/2$, subjected to additional forces of viscous friction with a Rayleigh dissipative function $F_N = \gamma (f^{\cdot})^2/2$. The coefficients α, β and γ satisfy the conditions

$$\alpha = \alpha_0 (N + o(N)), \quad \beta = \beta_0 (N + o(N)), \quad \gamma = \gamma_0 (N + o(N)), \tag{1.2}$$

where $\alpha_0, \beta_0, \gamma_0$ are non-negative real numbers, N is a positive parameter which will ultimately be made to approach $+\infty$. The added term in the expression for the kinetic energy T_N represents anisotropy of the mass distribution (such as apparent additional masses and moments of inertia in the problem of a solid moving in a liquid). The additional potential $\beta f^2/2$ sets up a force field directed towards the surface $f = 0$. We note in addition that the dissipative forces do not perform any work when the motion is confirmed to the surface $f = 0$.

We write the equations of motion

$$\left(\frac{\partial L_N}{\partial x^{\cdot}}\right)^{\cdot} - \frac{\partial L_N}{\partial x} = -\frac{\partial F_N}{\partial x^{\cdot}}, \quad L_N = T_N - V_N \tag{1.3}$$

and consider their solutions with initial conditions $x(0) = x_0$, $x^{\cdot}(0) = x_0^{\cdot}$, such that

$$f(x_0) = 0, \quad f_x'(x_0)\, x_0^{\cdot} = 0 \tag{1.4}$$

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Theorem 1. Assume that a solution of Eqs.(1.1) with initial data $x_0, x_0^{\cdot}$ is defined in an open time interval containing $[0, T]$, and that not all the coefficients α_0, β_0 and γ_0 in (1.2) vanish. Then, for sufficiently large N, solutions of Eqs.(1.3) with the same initial data are defined in the interval $0 \leqslant t \leqslant T$, and as $N \to \infty$ they tend to solutions of system (1.1).

If $\alpha_0 > 0$, Theorem 1 may be proved by the method used in /1/. In the special case when $\gamma_0 = 0$, the theorem actually follows from the results of that paper. Let $\alpha_0 = 0$, but $\beta_0 > 0$. Then system (1.3) is singularly perturbed, and the theorem may be proved by applying the well-known Tikhonov-Gradshtein Theorem. Note that when $\gamma_0 = 0$ this result follows from a more general result on the realization of non-holonomic constraints by means of anisotropic friction, which goes back to Carathéodory (see /2, 3/, and also the appendix to /4/). In the case $\alpha_0 = \beta_0 = 0$ but $\gamma_0 > 0$, Theorem 1 was put forward by Courant and proved in /5/.

2. A system with unilateral constraints. Let us apply the ideas set out in Sect.1 to the problem of realizing a unilateral constraint $f(x) \geqslant 0$. We again consider a solution of Eqs.(1.1) with initial data $x_0, x_0^{\cdot}$ satisfying (1.4) and let $\lambda(t)$ be the values of λ along this solution. Now consider a motion $x(t)$ with a unilateral constraint $f(x) \geqslant 0$ and the previous initial data. It is known that if $\lambda(t) < 0$ for all t, then the point $x(t)$ remains constantly on the surface $f(x) = 0$. But if from some time τ onwards $\lambda(t)$ takes positive values, then at a time τ the point $x(t)$ leaves the surface $f(x) = 0$ and at $t > \tau$ the dynamics of the system is described by the usual Lagrange equations with Lagrangian $L = T - V$ /6/ (note that the case $\tau = 0$ is not excluded).

To proceed by a passage to the limit from the dynamics of the free system to that of the system with the unilateral constraint $f \geqslant 0$, we consider motion in a force field with potential

$$V_N = V + Nf^2, \quad \text{if} \quad f < 0, \quad \text{and} \quad V_N = V, \quad \text{if} \quad f \geqslant 0 \tag{2.1}$$

The kinetic energy T is left unchanged. The motion of the system is determined by the Lagrange equations

$$(T_{x^{\cdot}}')^{\cdot} - T_x' = -V_{Nx}' \tag{2.2}$$

The generalized force $-(V_N)'$ is only piecewise-smooth, but it can be verified that Eqs.(2.2) satisfy the assumptions of the existence and uniqueness theorem.

Theorem 2. Let $x(t)$ be a motion of the system with unilateral constraint $f(x) \geqslant 0$ and initial data (1.4), defined in an interval $[0, T_*]$. Assume that there is at most one $\tau \in [0, T_*)$ such that $f(x(t)) = 0$ for $0 \leqslant t \leqslant \tau$ and $f(x(t)) > 0$ for $\tau < t \leqslant T < T_*$. If $x_N(t)$ is the solution of system (2.2) with initial data (1.4), then for sufficiently large N it is defined for $0 \leqslant t \leqslant T$ and in that interval tends to $x(t)$ as $N \to \infty$.

This assertion can be proved by the methods used in /5/.

It should be noted that Theorem 2 is true both for $\tau = 0$ (the point $x(t)$ immediately leaves the surface $f = 0$) and for $\tau > T_*$ $x(t)$ moves on the surface $f = 0$).

Rather than present the cumbersome formal proof of Theorem 2, we will illustrate it by an example, and also discuss the possibility of using the effect of apparent additional masses and anisotropic viscous friction to realize unilateral constraints. Consider the motion of a point of unit mass in the x, y plane, subjected in the left half-plane ($x \leqslant 0$) to the action of a force with components $(0, -g)$, $g = \text{const} > 0$, and in the right half-plane ($x > 0$) to the action of a force with components $(0, g)$. This force is irrotational, but not continuous. If at time $t = t_*$ the point was on the vertical line $x = 0$, then its state at time $t = t_*$ is taken as the initial state for determining its subsequent motion in the other half-plane. Consider the motion with a unilateral constraint $y \geqslant 0$. Suppose that the initial state of the point at $t = 0$ is

$$x = -1, \quad y = 0, \quad x^{\cdot} = 1, \quad y^{\cdot} = 0 \tag{2.3}$$

Then the law of motion is

$$x(t) = t - 1; \quad y(t) = 0, \quad t \leqslant 1, \quad y(t) = g(t-1)^2/2, \quad t > 1 \tag{2.4}$$

At time $t = 1$ the point leaves the constraint.

We now free the system from the constraint, replacing the action of the latter by an elastic force with components $(0, -Ny)$, $y \leqslant 0$, vanishing in the upper half-plane. Then the solution of the new equations of motion with initial data (2.3) at $t \leqslant 1$ is

$$x(t) = t - 1, \quad y(t) = gN^{-1}(\cos \sqrt{N}t - 1)$$

Consequently, at times $0 \leqslant t \leqslant 1$ the point is confined to a narrow strip $-2g/N \leqslant y \leqslant 0$, performing oscillations at a high frequency $\sqrt{N}$ (Fig.1, the solid curve). Then, at some time $\tau = 1 + O(N^{-1/2})$, it reaches the axis $y = 0$, with $y(\tau) = O(N^{-1})$, $y^{\cdot}(\tau) = O(N^{-1/2})$. At $t > \tau$ the point will describe a parabola in the upper half-plane. In the limit $(N \to \infty)$ the motion of the point is described precisely by formulae (2.4).

We will now consider another mechanism for realizing the constraint: if the point enters the lower half-plane, its kinetic energy receives an increment $Ny^{\cdot 2}/2$. Solving the equations of motion of the mechanical system with initial data (2.3), we obtain

$$x(t) = t - 1; \quad y(t) = -\frac{gt^2}{2(N+1)}, \quad t \leqslant 1 \tag{2.5}$$
$$y(t) = \frac{g(t^2 - 4t + 2)}{2(N+1)}, \quad 1 \leqslant t \leqslant 2 + \sqrt{2}$$

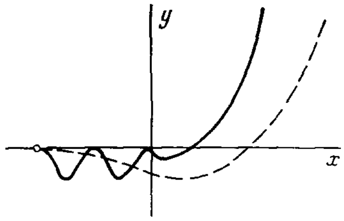

Fig.1

Consequently, if $0 < t < 2 + \sqrt{2}$, then $y(t) < 0$ (Fig.1, the dashed curve). At $t > 2 + \sqrt{2}$ the point will describe a parabola in the upper half-plane. Note that the time at which the point crosses the horizontal line $y = 0$ $(t = 2 + \sqrt{2})$ is independent of N. As $N \to \infty$ the solution (2.5) tends to a solution

$$x(t) = t - 1; \quad y(t) = 0, \quad t \leqslant 2 + \sqrt{2}, \quad y(t) = g(t - 2 - \sqrt{2})^2/2, \quad t > 2 + \sqrt{2} \tag{2.6}$$

which is different from (2.4). The extra time taken by the moving point to leave the constraint is due to its additional inertia.

Finally, let us consider one more case: a point initially in the lower half-plane is subjected to a viscous friction force with components $(0, -Ny^{\cdot})$. Solving the equations of motion with initial data (2.3), we find that at $0 < t < \tau$, $\tau = 2 + O(N^{-1})$, the point moves in the lower half-plane and it crosses the horizontal line at time τ. Its trajectory is qualitatively similar to the dashed curve in Fig.1. At $t > \tau$ the point again describes a parabola. If we now let N go to $+\infty$ the limiting motion is

$$x(t) = t - 1; \quad y(t) = 0, \quad t \leqslant 2, \quad y(t) = g(t - 2)^2/2, \quad t > 2 \tag{2.7}$$

which differs from both (2.4) and (2.5). But if the force $(0, -Ny^{\cdot})$ is put equal to zero (then $y^{\cdot} > 0$) and N goes to infinity, we obtain the classical solution (2.4).

Thus, by introducing an elastic force field and then increasing the coefficient of viscosity to infinity we obtain the classical model of motion with unilateral constraints. Introduction of apparent additional masses and anisotropic viscous friction leads to the classical models, but with a delay in the time necessary to leave the constraint.

One cannot assert that motions (2.6) and (2.7) bear any relation to reality. They are motions in well-defined models of systems with unilateral constraints, whose choice depends essentially on the specific physical methods used to realize the constraints. It should be borne in mind that in reality one can have combinations of various effects, also leading in the limit to a constrained motion (see, e.g., Sect.1). The order of magnitude of the additional forces then plays an important role.

For example, consider the case of a point subjected in the lower half-plane $(y < 0)$ to the simultaneous action of a friction force $(0, -Ny^{\cdot})$ and an elastic force $(0, -c^2Ny)$. Replacing the unilateral constraint $y \geqslant 0$ by linear fields of elastic and dissipative forces is physically equivalent to the consideration of a Kelvin-Voigt medium. Calculations show that the motion with initial data (2.3) as $N \to \infty$ is again characterized by a delay in the time necessary to leave the constraint. But if the friction force is replaced by a dissipative force $(0, -k\sqrt{N}y^{\cdot})$, where $k = \text{const} > 0$ and $c^2 > k^2$, then as $N \to \infty$ one obtains the classical solution (2.4).

Note that Theorem 2 is also true in case the potential V_N is replaced by a potential $V_N^* = V + \exp(-Nf)$. This remark may prove useful in analytical studies, since the function V_N^* is infinitely differentiable.

3. Motions with collisions. Let $x(t)$ be the motion of a system with a non-retaining constraint $f(x) \geqslant 0$, where $f(x(t)) > 0$ for $t_1 < t < t_2$. If $f(x(t_2)) = 0$ and the velocity $x^{\cdot}(t_2)$ is not in the tangent plane to the surface $f(x) = 0$ at $x = x(t_2)$, then at time $t = t_2$ a collision effect will take place. The motion may continue at times $t > t_2$ in various ways, depending on the hypothesis adopted as to the physical nature of the collision (absolutely elastic, inelastic, etc.). In order to verify these hypotheses, and also to ascertain the limits of their applicability, we drop the constraint $f \geqslant 0$ and consider a force field in the region $f(x) < 0$ with potential $V_N = V + c^2Nf^2/2, c > 0$, assuming that the system moves in that region subject to viscous friction forces with a Rayleigh dissipative function $F_N = k\sqrt{N}(f^{\cdot})^2$, $k = \text{const} \geqslant 0$ (see Sect.1).

We shall assume that at a time $t = 0$ the system is in the position $x = (x_1, \ldots, x_n) = 0$ and has a velocity $x^{\cdot} = (v_1, \ldots, v_n)$, and moreover

$$\sum \partial f/\partial x_i|_{x=0} v_i < 0 \tag{3.1}$$

At $t > 0$ the point $x(t)$ enters the region $f < 0$ and the system is subjected to the action of additional irrotational and dissipative forces. It turns out that if N is large then, in a short time interval (of the order of $1/\sqrt{N}$) the point $x(t)$ will hit the boundary $f = 0$ at a point near $x = 0$; the tangential components of the velocity at the time the point enters and leaves the region $f(x) < 0$ remain practically unchanged, and the normal component of the velocity (in the metric determined by the kinetic energy) at the time of exit may be expressed in terms of the normal component at the time of entry and the coefficients k and c. As $N \to \infty$ we obtain the laws of inelastic collision (as $k \to 0$ they become the laws of elastic reflection).

To implement this idea, it is convenient to transform to special coordinates $x_1, \ldots, x_n$ in the neighbourhood of the point $x = 0$, in terms of which $f(x) \equiv x_1$ and the kinetic energy has the form

$$T = \frac{1}{2} g_{11} x_1^{\cdot 2} + \frac{1}{2} \sum_{i, j=2}^{n} g_{ij}(x) x_i^{\cdot} x_j^{\cdot}$$

Such coordinates always exist (in Riemannian geometry they are known as semigeodesic coordinates /7/). In terms of them, the tangential component of the velocity $x^{\cdot}$ is determined by the components $x_2^{\cdot}, \ldots, x_n^{\cdot}$, and the normal component by the derivative $x_1^{\cdot}$. Relation (3.1) takes the simpler form $x_1^{\cdot}(0) < 0$.

Theorem 3. Assume that $k^2 < c^2 g_{11}(0)$. Then there exists a time

$$\delta_N = \pi g_{11}(0)/(\omega \sqrt{N}) + o(N^{-1/2}), \quad \omega = \sqrt{c^2 g_{11}(0) - k^2} > 0$$

such that $x_1(\delta_N) = 0$, $x_i(\delta_N) = O(N^{-1/2})$ $(i > 1)$ and the following limit relations hold:

$$\lim_{N\to\infty} x_1^{\cdot}(\delta_N) = -e x_1^{\cdot}(0), \quad e = \exp(-\pi k/\omega) \tag{3.2}$$

$$\lim_{N\to\infty} x_i^{\cdot}(\delta_N) = x_i^{\cdot}(0), \quad i > 1$$

The number $e < 1$ is known in the theory of stereomechanical impact as the coefficient of restitution /8/. If $k = 0$ (no dissipation), then (3.2) constitute the conditions for an absolutely elastic collision. We now outline the proof of Theorem 3.

We write the equations of motion for those values of $t > 0$ at which $x_1(t) \leqslant 0$:

$$\begin{aligned} &x_1^{\cdot\cdot} + \sum \Gamma_{jk}^1 x_j^{\cdot} x_k^{\cdot} = -g^{11}\partial V/\partial x_1 - 2k\sqrt{N}\, g^{11} x_1^{\cdot} - c^2 N g^{11} x_1 \\ &x_i^{\cdot\cdot} + \sum \Gamma_{jk}^i x_j^{\cdot} x_k^{\cdot} = -\sum g^{is}\partial V/\partial x_s \quad (i > 1) \end{aligned} \tag{3.3}$$

Here Γ_{jk}^i are the Christoffel symbols of the metric g_{ij}, and g^{is} are the elements of the matrix inverse to $\| g_{ij} \|$. Now transform Eqs.(3.3) to a new time variable $\tau = \sqrt{N}t$, denoting differentiation with respect to τ by a prime:

$$\begin{aligned} &x_1'' + 2k g^{11} x_1' + c^2 g^{11} x_1 + \sum \Gamma_{jk}^1 x_j' x_k' = -\varepsilon^2 g^{11}\partial V/\partial x_1 \\ &x_i'' + \sum \Gamma_{jk}^i x_j' x_k' = -\varepsilon^2 \sum g^{is}\partial V/\partial x_s \quad (i > 1); \quad \varepsilon = N^{-1/2} \end{aligned} \tag{3.4}$$

Letting ε be a small parameter, we seek solutions of system (3.4) in the form

$$x_i(\tau) = \varepsilon \xi_i(\tau) + o(\varepsilon), \quad i = 1, \ldots, n \tag{3.5}$$

The coefficients ξ_i satisfy the equations

$$\xi_1'' + 2kG\xi_1' + c^2 G\xi_1 = 0, \quad \xi_i'' = 0 \ (i > 1); \quad G = g^{11}(0) > 0$$

These equations have the following particular solutions:

$$\begin{aligned} &\xi_1 = x_1^{\cdot}(0)\exp(-kG\tau)\,\omega^{-1}\sin\omega\tau \\ &\omega = \sqrt{c^2 G - k^2 G^2}, \quad \xi_i = x_i^{\cdot}(0)\tau \quad (i > 1) \end{aligned} \tag{3.6}$$

Since ε occurs regularly in the right-hand terms of Eq.(3.4), we can apply Poincaré's Theorem /9/, according to which, for small values of ε system (3.4) has solutions of the type (3.5) in any finite interval of the new time variable τ. By (3.6), the function $x_1(\tau)$ must have a zero in the interval $(0, 2\pi)$. Returning to the previous time variable t, we obtain

$$\begin{aligned} &x_1(t) = x_1^{\cdot}(0)\exp(-kG\sqrt{N}t)(\omega\sqrt{N})^{-1}\sin\omega\sqrt{N}t + \\ &\quad o(N^{-1/2}) \\ &x_i(t) = x_i^{\cdot}(0)\,t + o(N^{-1/2}) \end{aligned}$$

These formulae imply (3.2).

Theorem 3 expands the limits of applicability of Theorem 2. Let $x(t)$ be the motion of a system with unilateral constraint $f(x) \geqslant 0$, beginning at the boundary $f(x) = 0$. Assume that in the interval $0 \leqslant t \leqslant T$ there are finitely many times $\tau_1, \ldots, \tau_s$ at which the system "collides" with the constraint, and that the velocities at collision $x^{\cdot}(\tau_i - 0)$ are transverse to the tangent planes to the surface $f(x) = 0$ at the points $x(\tau_i)$. Theorem 2 states that the motions of the free system in a field with potential V_N, with the same initial data, tend to the motion $x(t)$ in the interval $0 \leqslant t < \tau_1$ as $N \to \infty$. Theorem 3 (in the special case $k = 0$) implies that this limiting relationship holds over the entire interval $0 \leqslant t \leqslant T$. Furthermore, if $k = 0$ Theorem 3 is also true for the potential $V_N^* = V + \exp(-Nf)$ (see Sect. 2).

As an illustrative example, let us consider the inertial motion of a point inside a circle of radius R, on the assumption that the coefficient of restitution e on impact is constant and close to unity. Since the tangential component of the velocity is constant, one has an integral of areas. Consequently, at the collision times $v \cos \alpha = u = \text{const}$, where v is the velocity of the point and α the angle between the velocity vector and the tangent to the circle.

Let us investigate the evolution of v as a function of time. To this end (by Theorem 3), we replace the unilateral constraint by a field of elastic forces and by dissipative forces with coefficient of friction $2k/\sqrt{N}$, where k is small. If $k = 0$ we have an unperturbed integrable problem. Its phase space is partitioned into invariant tori with conditionally periodic motions. Averaging the perturbed equations over these tori and letting $N \to \infty$, we can obtain an evolution equation for the velocity:

$$v^{\cdot} = (e - 1)(2R)^{-1} v \sqrt{v^2 - u^2}$$

Integrating, we obtain

$$v = u / \cos[u(e - 1)(2R)^{-1} t + \alpha_0]$$

where α_0 is the value of the angle α at time $t = 0$. Consequently, at time $t > 2R/[u(1 - e)]$ the representative point of the averaged system will slide along a circle of radius R with velocity u.

4. **Stability of periodic oscillations with collisions.** We shall now demonstrate the efficacy of the proposed method in the theory of systems with unilateral constraints, considering as an example the stability of periodic oscillatory modes with collisions. Consider the motion of a material point of mass m in a vertical plane with Cartesian coordinates x, y (the y axis pointing upwards), never falling below the curve $y = f(x)$, where f is a smooth function such that $f(0) = 0$, $f'(0) = 0$, $f''(0) > 0$. This problem admits of a family of periodic motions, in which the point m oscillates constantly on the y axis. As parameter we take the velocity v of the point at its time of collision with the curve. A criterion was evolved in /10/ for this solution to be elliptic (g is the acceleration due to gravity):

$$f''(0) < gv^{-2} \tag{4.1}$$

We now replace the unilateral constraint $y \geqslant f(x)$ by an elastic force field with potential

$$V_N = mgy, \quad y \geqslant f(x); \quad V_N = mgy + mN(y - f(x))^2/2, \quad y < f(x)$$

The problem of the motion of the point m in a field with potential V_N has a family of T-periodic solutions:

$$\begin{aligned} &x_0(t) \equiv 0, \quad 0 \leqslant t \leqslant T \\ &y_0(t) = gN^{-1}(\cos\sqrt{N}t - 1) - vN^{-1/2}\sin\sqrt{N}t, \quad 0 \leqslant t < \tau \\ &y_0(t) = vt - gt^2/2, \quad \tau \leqslant t \leqslant \tau + 2vg^{-1} \\ &\tau = 2N^{-1/2}\operatorname{arcctg}(-gv^{-1}N^{-1/2}), \quad T = \tau + 2vg^{-1} \end{aligned} \tag{4.2}$$

where $v > 0$ is the velocity of m in the position $x = y = 0$. The equation for the x-coordinate of m is

$$x^{\cdot\cdot} = 0, \quad y > f; \quad x^{\cdot\cdot} = Nf'(x)(y - f), \quad y < f$$

The variational equation for the periodic solution (4.2) is

$$\begin{aligned} &(\delta x)^{\cdot\cdot} + p(t)\,\delta x = 0; \quad p(t) = -Nf''(0)\,y_0(t), \quad 0 \leqslant t \leqslant \tau; \\ &p(t) = 0, \quad \tau \leqslant t \leqslant T \end{aligned} \tag{4.3}$$

Since $p(t) \geqslant 0$, the conditions for stability of the trivial solution of Eq.(4.3) may be determined by the Lyapunov integral test /11/:

$$T\int_0^T p(t)\,dt \leqslant 4 \tag{4.4}$$

Calculation gives the following sufficient condition for stability:

$$gf''(0)\,[g^{-1}v + N^{-1/2}\operatorname{arcctg}(-gv^{-1}N^{-1/2})]^2 \leqslant 1$$

For large N values this expression can be simplified:

$$f''(0) \leqslant gv^{-2} - \pi g^2 v^{-3} N^{-1/2} + o(N^{-1/2}) \tag{4.5}$$

In the limit as $N \to \infty$ we obtain the well-known condition (4.1). Inequality (4.5) shows that replacing the unilateral constraint by an elastic force field of high stiffness (which is a better approximation to reality) may cause the vertical oscillations of the point to become unstable.

It is an interesting fact that as $N \to \infty$ the stability condition (4.5) approaches the criterion for stability in the linear approximation (4.1). This observation leads to Zhukovskii's result, according to which the constant on the right-hand side of Lyapunov's inequality (4.4) cannot be increased /12/.

Let us consider another similar problem, concerning the stability of the two-linked periodic trajectory of a Birkhoff billiard ball (Fig.2); a point is moving on a segment l, periodically and elastically recoiling from a curve. Let R_1 and R_2 denote the radii of curvature of the curve at the endpoints of the segment, $R_1 \leqslant R_2$. We again introduce an elastic force field, and the result is a variational equation similar to (4.3):

Fig.2

$$\begin{gathered}\xi^{\cdot\cdot} + p(t)\xi = 0 \\ p(t) = vR_1^{-1}\sqrt{N}\sin\sqrt{N}t, \quad 0 \leqslant t \leqslant \pi N^{-1/2} \\ p(t) = vR_2^{-1}\sqrt{N}\sin\sqrt{N}(t+\tau), \\ \tau \leqslant t \leqslant \tau + \pi N^{-1/2}, \quad \tau = \pi N^{-1/2} + lv^{-1} \\ p(t) = 0, \quad \pi N^{-1/2} \leqslant t \leqslant \tau, \quad \tau + \pi N^{-1/2} \leqslant t \leqslant \tau + \pi N^{-1/2} + lv^{-1}\end{gathered} \tag{4.6}$$

Here v is the constant velocity of the point inside the billiard table; the values of the function $p(t)$ are prescribed over its period. Applying the Lyapunov integral test (4.4) and then letting $N \to \infty$, we obtain a sufficient condition for stability (to a first approximation) of the two-linked trajectory

$$l \leqslant R_1 R_2/(R_1 + R_2) \tag{4.7}$$

In the general case, however, this condition is not necessary. The condition for the stability of the two-linked periodic trajectory is (see, e.g., /13/)

$$l < R_1 \quad \text{or} \quad R_2 < l < R_1 + R_2 \tag{4.8}$$

Condition (4.8) is identical with (4.7) if $R_2 \to \infty$. In that case the function $p(t)$ in Eq.(4.6) has only one short interval in which it is positive (as in Eq.(4.3)).

Condition (4.8) may be derived by means of Lyapunov's general method of analysing the stability of the trivial solution of the second-order Eq.(4.6) /11/. The stability criterion is the inequality $|a| < 2$ for Lyapunov's constant

$$a = 2 - I_1 + I_2 - I_3 + \ldots, \quad I_1 = T\int_0^T p(t_1)\,dt_1$$

$$I_k = \int_0^T dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{k-1}} dt_k (T - t_1 + t_k)(t_1 - t_2)\ldots(t_{k-1} - t_k)\,p(t_1)\ldots p(t_k)$$

Let us assume first of all that $p(t) = R_1^{-1}\sqrt{N}\sin\sqrt{N}\,t,\ 0 \leqslant t \leqslant \pi N^{-1/2}$, and that for $\pi N^{-1/2} \leqslant t \leqslant T$ the function $p(t)$ vanishes (we may assume without loss of generality that the velocity v equals unity). It is clear that $I_1 = 2TR_1^{-1}$, and the other integrals satisfy the estimate

$$|I_k| \leqslant c\,(N^{-1/2})^{k-1}, \quad k \geqslant 2, \quad c = \text{const} > 0$$

since the integration is performed only over a small domain in the space of the variables $t_1, \ldots, t_k$ and the differences $t_i - t_j$ are of the order of $N^{-1/2}$. Letting $N \to \infty$ we have $T - 2l$ and the inequality $|a| \leqslant 2$ becomes $l \leqslant R_1$ (corresponding to the case $R_2 = \infty$ in (4.8)).

We now consider the general case of Eq.(4.6). It can be verified that

$$\lim_{N\to\infty} I_1 = 4l(R_1^{-1} + R_2^{-1}), \quad \lim_{N\to\infty} I_2 = 4R_1^{-1}R_2^{-1}l^2$$

while the integrals $I_k,\ k \geqslant 3$, satisfy the estimates $|I_k| \leqslant c(N^{-1/2})^{k-2},\ c > 0$. As $N \to \infty$ the inequality $|a| < 2$ becomes

$$|1 - 2l(R_1^{-1} + R_2^{-1}) + 2R_1^{-1}R_2^{-1}l^2| < 1$$

This is clearly equivalent to (4.8).

In the same way, one can obtain a condition for the stability of the periodic motion of a point between concave walls in the gravitational field (Fig.2). Let $v_1 (v_2)$ be the velocity of the point in its lowest (uppermost) position. Since we are considering motion with collisions, it is assumed that $v_2^2 = v_1^2 - 2gl \geqslant 0$. The stability condition in the linear approximation is

$$\left|1 - \frac{2(v_1 - v_2)}{g}\left(\frac{v_1}{R_1} + \frac{v_2}{R_2}\right) + \frac{2(v_1 - v_2)^2}{g^2}\,\frac{v_1 v_2}{R_1 R_2}\right| < 1 \tag{4.9}$$

If $v_2 = 0$, this condition is equivalent to (4.1). Now let g tend to zero. In the limit we obtain the inertial motion of a point between two immovable walls. In that case $v_i \to v$ $(i = 1,2)$ and $v_1 - v_2 = gl/v + o(g)$. Letting $g \to 0$ in inequality (4.9), we obtain the disired stability condition (4.8).

To conclude this section, we note that the coefficient $p(t)$ in Eq.(4.6) is closely connected with the Dirac δ-function. Indeed, for any continuous function $f(x)$,

$$\lim_{\lambda\to\infty} \lambda \int_0^{\pi/\lambda} f(x)\sin(\lambda x)\,dx = 2f(0)$$

Thus, as $N \to \infty$ we obtain the formal relation

$$p(t) = \sum_{n=-\infty}^{\infty} 2v\left[R_1^{-1}\delta\left(\frac{nl}{v}t\right) + R_2^{-1}\delta\left(\frac{nl}{v}t + l\right)\right]$$

Substituting this expression formally into the expressions for the integrals I_k, and using the inequality $|a| < 2$ for the Lyapunov constant, we obtain the stability condition (4.8).

5. Parabolic billiards.

In /10/ KAM-theory was used to investigate the non-linear stability of periodic up-and-down jumps of a point in the gravitational field above a curve $y = f(x)$. In particular, it was shown that for a parabola $(f(x) = x^2/2a,\ a > 0)$, satisfying condition (4.1), these solutions are orbitally Lyapunov-stable. It turns out that this result follows quite easily from a stronger assertion, concerning the integrability of the parabolic billiards problem.

To prove this, let us consider the problem of a material point sliding along a smooth paraboloid:

$$2y + b = x^2/(a+b) + z^2/b;\quad a,\ b > 0 \tag{5.1}$$

As observed by Painlevé, this system with two degrees of freedom has an additional first integral. Chaplygin reduced the integration of the equations of motion of inversion of Abelian integrals /14/. To do so, he introduced parabolic coordinates $v,\ w\ (w \geqslant a \geqslant v \geqslant 0)$ by the formulae

$$ax^2 = (a+b)(a-v)(w-a),\quad 2y = v + w - (b+a),\quad az^2 = bvw \tag{5.2}$$

and derived equations for the variation of v, w:

$$v^{\cdot} = \frac{4}{m(w-v)}\left[\frac{(\alpha v^2 - \beta v - \gamma)(a-v)v}{v+b}\right]^{1/2}$$
$$w^{\cdot} = \frac{4}{m(w-v)}\left[\frac{(-\alpha w^2 + \beta w + \gamma)(w-a)w}{w+b}\right]^{1/2} \tag{5.3}$$

$$\alpha = \frac{m^2 g}{4},\quad \beta = \frac{mh}{2} + \frac{m^2(a+b)g}{4}$$

Here m is the mass of the point, g is the acceleration due to gravity, h is the total energy of the point, and γ is a constant of integration.

Now let the parameter b tend to zero. Then the paraboloid (5.1) becomes the region above the parabola $y = x^2/(2a)$ in the vertical plane $z = 0$, and the motion of the point along the paraboloid becomes free fall along the parabola with absolutely elastic collisions. A limit procedure of this type was apparently first discussed by Birkhoff /15/, who noted that the geodesics on the triaxial ellipsoid are transformed into the trajectories of a billiard ball in an elliptic field as one of the axes is reduced to zero. Since the Painlevé-Chaplygin problem is integrable for all values of $b > 0$, the limiting problem of parabolic billiards is also integrable. There is not much point in evaluating the additional integral, since relations (5.2) and (5.3) (in which one must put $b = 0$) determine the law of motion of the point over the entire time axis.

Formulae (5.2) and (5.3) enable one to present a complete qualitative analysis of the motion of a material point with unilateral constraint $y \geqslant x^2/2a$ (see /14/). To this end, we consider the polynomial $F(z) = \alpha z^2 - \beta z - \gamma$, which always has real roots; denote the latter by $z_1, z_2\ (z_1 \leqslant z_2)$. Since $F(v) \geqslant 0$, $F(w) \leqslant 0$, $w \geqslant a \geqslant v \geqslant 0$, it follows that $z_2 \geqslant w > z_1 \geqslant v$.

We distinguish two cases: 1) $z_2 > a \geqslant z_1$, 2) $z_2 \geqslant z_1 \geqslant a$. In the first case $z_2 \geqslant w \geqslant a,\ z_1 \geqslant v \geqslant 0$. The trajectory of the point m is confined to the interior of a curvilinear quadrilateral

$$x^2/a \leqslant 2y \leqslant z_1 + x^2/(a - z_1),\quad 2y \leqslant z_2 + x^2/(a - z_1) \tag{5.4}$$

bounded by confocal parabolas (see Fig.3a). In the second case we have $z_2 \geqslant w \geqslant z_1,\ a \geqslant v \geqslant 0$, and the trajectory of m is confined to the interior of the quadrilateral

$$z_1 + x^2/(a - z_1) \leqslant 2y \leqslant z_2 + x^2/(a - z_2),\quad x^2/a \leqslant 2y \tag{5.5}$$

illustrated in Fig.3b. In the general case, the trajectory of m fills the quadrilaterals densely.

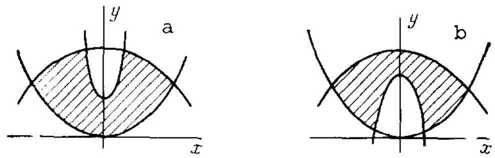

Fig.3

The trajectories of the vertical oscillations of m correspond to the case $F(a)=0$, and therefore $\gamma=\alpha a^2-\beta a$. If $2a\alpha>\beta$ (equivalent to (4.1)), then small perturbations of the constant first integrals h and γ lead to the first case and the quadrilateral (5.4) lies near the y axis. In that case the periodic motion is stable. But if $F(a)=0$ and $2a\alpha<\beta$, then $z_1=a$ and the quadrilateral (5.5) degenerates to a lens-shaped figure

$$x^2/a\leqslant 2y\leqslant z_2+x^2/(a-z_2),\quad z_2>a \tag{5.6}$$

In this case the vertical periodic oscillations are unstable. They are of hyperbolic type and consequently possess asymptotic trajectories. These asymptotic motions are in fact doubly asymptotic and exhibit a remarkable property: after each recoil, the trajectory of the point m passes through the focus of the parabola $y=x^2/(2a)$ (which is, of course, in the interior of the region (5.6)). Moreover, the time between consecutive passages through the focus is a constant: $2g^{-1}\sqrt{2hm^{-1}}$. This is of course the period of the vertical oscillations of m.

If $z_1=z_2>a$, the point m moves along the parabola

$$2y=z_1+x^2/(a-z_1),\quad y\geqslant x^2/(2a) \tag{5.7}$$

recoiling periodically from the original parabola $y=x^2/(2a)$, which is confocal with (5.7). All these periodic motions have the same period $4\sqrt{a/g}$. In the special case $z_1=z_2=a$ the solution (5.7) degenerates to periodic vertical jumps of the point m to the height $a/2$ (the distance from the focus of the parabola). These oscillations are stable.

This analysis furnishes a complete and graphic description of all non-degenerate motions of the point m. Let the energy be $h=0$. Then the point m occupies the lowest stable equilibrium position. Now increase h. At small $h>0$ there appear two distinct families of Lyapunov non-degenerate periodic motions: vertical jumps and sliding along the parabola. The solutions of the second family exist for all $h>0$ and are all stable (as the limiting case of solutions of type 1). The solutions of the first family also exist for all h, but when $h=mga$ (when the height of the jump equals the distance to the focus of the parabola) the multipliers become equal to unity. This is a bifurcation point: when $h>mga$ there appears yet another family of stable periodic oscillations (5.7), and the vertical periodic jumps become unstable.

6. Harmonic oscillator and elliptic billiards.

We now consider the motion of a material point along the smooth surface of a triaxial ellipsoid under the action of an elastic force directed towards (or away from) the centre of the ellipsoid. This problem was integrated by Jacobi, who used elliptic coordinates /16/. If we let one of the axes of the ellipsoid go to zero, Jacobi's problem becomes the problem of the oscillations of a harmonic oscillator confined to the interior of an ellipse. If the coefficient of elasticity is zero, we obtain Birkhoff's elliptic billiards game. The dynamics of a harmonic oscillator inside an ellipse may be investigated by the method of Sect.5, using separation of variables - elliptic coordinates in the plane.

For example, let us determine the conditions for the stability of the periodic oscillations of the oscillator, under which the point remains constantly on one of the axes of symmetry of the ellipse. Let a, b be the semi-axes of the ellipse ($b\leqslant a$), c the coefficient of elasticity, and h the total mechanical energy of the oscillator. It turns out that if $c\geqslant 0$, the motion of the point along the minor (major) axis of the ellipse is stable if and only if

$$h\geqslant 0\quad (0\leqslant h\leqslant {}^1/_2ca^2) \tag{6.1}$$

But if $c<0$, these conditions become

$$h\geqslant {}^1/_2|c|(a^2-b^2)\ (-{}^1/_2|c|a^2\leqslant h\leqslant -{}^1/_2|c|(a^2-b^2)) \tag{6.2}$$

The second of these conditions has a simple geometrical meaning: the amplitude of the periodic oscillations of the point does not exceed the distance from the end of the major semi-axis to the nearest focus. As the amplitudes increases, this solution destablilizes, ultimately becoming hyperbolic. The trajectories through the focus of the ellipse have a curious property: in equal intervals of time, the point alternately passes through the foci infinitely many times. This property is also valid for trajectories that are not tangent to the boundaries of the billiard.

Let $c < 0$. Let us deform the ellipse in such a way that one of its foci remains fixed, while the other goes to infinity, and moreover $(a - \sqrt{a^2 - b^2}) \rightarrow \text{const}$. As a result, the ellipse degenerates to a parabola. If at the same time we also reduce the coefficient of elasticity, in such a way that $|c| a \rightarrow g$, the problem of a harmonic oscillator inside an ellipse becomes the parabolic billiards problem investigated in Sect.5. It can be shown that in this limiting process the second stability condition of (6.2) becomes the well-known condition (4.1).

The above results furnish stability conditions (in the linear approximation) for oscillations of a plane harmonic oscillator situated half-way between two convex surfaces of the same curvature. According to Sect.4, stability in the linear approximation depends only on the curvature of these curves at the endpoints of the rectilinear trajectory, but not on their shape.

Let l be the length of the periodic trajectory of the oscillator, and R the radius of curvature of the curves at the endpoints of the trajectory. We consider motion with collisions. If $c > 0$, the motion is stable only provided $l < 2R$ (see (6.1)). Comparing this result with inequalities (4.8), we see that the presence of an attractive elastic force does not affect the stability of oscillations with collisions. Now let $c < 0$. If $l < 2R$, the motion is stable when $4h > |c| l (R - l/2)$. But if $l > 2R$, the stability condition is expressed by the inequality $4h < |c| l (R - l/2)$. In the case of equality, $l = 2R$, the periodic oscillation is degenerate: its multipliers equal unity.

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